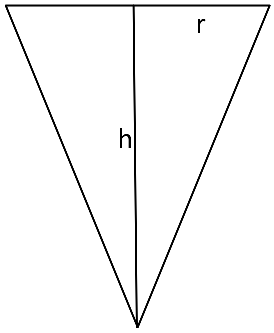
ESTABLISHING A MODEL – involving Finding Rates of Change, and Optimisation

1) An ice-cream cone of inside height 15 cm and inside uppermost radius 6cm is being filled with soft ice-cream at a rate of 20 cm^3/s. Find the rate at which the depth of ice-cream is rising at the moment when the depth is 5 cm.



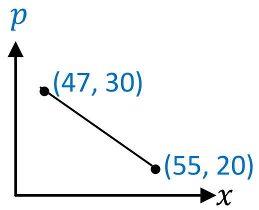
- Find $\frac{dh}{dt}$, given $\frac{dV}{dt} = 20\ cm^3/s$
- Use Chain Rule: $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$
- We need an expression for $\frac{dh}{dV}$, which we will get from the reciprocal of $\frac{dV}{dh}$.
- V = $\frac{1}{3}\pi r^2 h$. We need to rewrite V in terms of only h, not both r and h, so we need an equation that links r and h.
- The ice cream cone has a set radius of 6 cm and height of 15 cm, so the ratio of the radius and height of the ice cream in the cone will be the same as the ratio of the radius and height of the cone: $\frac{r}{h} = \frac{6}{15}$ => $15r = 6h => r = \frac{6h}{15} = \frac{2h}{5}$.
- Rewrite V in terms of only h: V = $\frac{1}{3}\pi(\frac{2h}{5})^2 h = \frac{4\pi h^3}{75}$
- $\frac{dV}{dh} = \frac{12\pi h^2}{75} = \frac{4\pi h^2}{25}$
- So the expression for $\frac{dh}{dt}$ is now $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{25}{4\pi h^2} \times 20 = \frac{125}{\pi h^2}$
- $\frac{dh}{dt}$ when $h = 5cm$ is $\frac{125}{\pi h^2} = \frac{125}{\pi(5)^2} = 1.59$ cm/s

2) Belinda has set up a small cottage industry. She is manufacturing and selling toddler's printed T-shirts. It cost her \$1000 to set up the business and each T-shirt costs her \$6 to make. If she sells the T-shirts for \$20, she sells 55 per week on average. If she charges \$30 per T-shirt, she only sells an average of 47 per week. What price should Belinda charge for her T-shirts in order to maximise her profit?

- Profit(P) = Revenue(R) – Cost(C)
- Number of T-shirts sold = x, price a T-shirt is sold for = p.
- Cost(C) = \$1000 + \$6 x number of T-shirts sold = 1000 + 6x
- Revenue(R) = number of T-shirts sold x price T-shirts are sold for = px
- P = $px - (1000 + 6x)$.
- To optimise P, we need to differentiate P with respect to **one** variable. There are currently 2 variables (p and x) in the expression for P, so we will need find an equation that links p and x, and later rewrite P in terms of one of the variables.
- When p=\$20, x=55 and when p=\$30, x=47. Treating these values like (x, y) coordinates and plotting them on an x-y axis to find an equation for a straight line, we get: $p = -\frac{5}{4}x + \frac{355}{4}$
- Rewrite expression for profit(P):

  P = $px - (1000 + 6x) = \left(-\frac{5}{4}x + \frac{355}{4}\right)x - (1000 + 6x)$

  $= -\frac{5}{4}x^2 + \frac{331}{4}x - 1000$



- P'=$-\frac{5}{2}x + \frac{331}{4}$. Set P' = 0 and solve for x.

  $-\frac{5}{2}x + \frac{331}{4} = 0 => x = 33.1$
- When $x = 33.1$, then $p = -\frac{5}{4}x + \frac{355}{4} = -\frac{5}{4}(33.1) + \frac{355}{4} = 47.375$
- Therefore, Belinda should charge her T-shirts \$47.38 to maximise her profits, even though she is selling less T-shirts.

Delta Ex 14.2 pg 142 – 143 Q7 – 17, Ex 11.2 pg 121 Q6 – 17, Ex 11.3 pg 124